

HALF-SPACE THEOREM, EMBEDDED MINIMAL ANNULI AND MINIMAL GRAPHS IN THE HEISENBERG GROUP

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ABSTRACT. We construct a one-parameter family of properly embedded minimal annuli in the Heisenberg group Nil_3 endowed with a left-invariant Riemannian metric. These annuli are not rotationally invariant. This family gives a vertical half-space theorem and proves that each complete minimal graph in Nil_3 is entire. Also, the sister surface of an entire minimal graph in Nil_3 is an entire constant mean curvature $\frac{1}{2}$ graph in $\mathbb{H}^2 \times \mathbb{R}$, and conversely. This gives a classification of all entire constant mean curvature $\frac{1}{2}$ graphs in $\mathbb{H}^2 \times \mathbb{R}$. Finally we construct properly embedded constant mean curvature $\frac{1}{2}$ annuli in $\mathbb{H}^2 \times \mathbb{R}$.

1. INTRODUCTION

This paper deals with global properties of minimal and constant mean curvature (CMC) surfaces in Riemannian homogeneous manifolds. Some interesting properties are the existence of a Hopf-type holomorphic quadratic differential (see [AR04], [AR05]) and Lawson-type local isometric correspondences, in particular between minimal surfaces in the Heisenberg group Nil_3 endowed with a left-invariant Riemannian metric and CMC $\frac{1}{2}$ surfaces in $\mathbb{H}^2 \times \mathbb{R}$ (see [Dan07]). Two surfaces related by this correspondence are called sister surfaces.

In this paper we first construct a one-parameter family of properly embedded minimal annuli that are somewhat analogous to catenoids of \mathbb{R}^3 with a “horizontal axis”.

Theorem (see theorem 5.6). *There exists a one-parameter family $(\mathcal{C}_\alpha)_{\alpha>0}$ of properly embedded minimal annuli in Nil_3 , called “horizontal catenoids”, having the following properties:*

- the annulus \mathcal{C}_α is not invariant by a one-parameter group of isometries,
- the intersection of \mathcal{C}_α and any vertical plane of equation $x_2 = c$ ($c \in \mathbb{R}$) is a non-empty closed embedded convex curve,
- the annulus \mathcal{C}_α is invariant by rotations of angle π around the x_1 , x_2 and x_3 axes and the x_2 -axis is contained in the “interior” of \mathcal{C}_α ,
- the annulus \mathcal{C}_α is conformally equivalent to $\mathbb{C} \setminus \{0\}$.

(The model we use for Nil_3 is described in section 2.)

Up to now, the only known examples of complete minimal surfaces in Nil_3 were surfaces invariant by a one-parameter group of isometries [FMP99], periodic surfaces [AR05] and entire graphs [Dan06]. The annuli we construct are the first

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non-trivial examples of annuli; they are very different from the rotationally invariant catenoids, which are of hyperbolic conformal type. The existence of annuli of parabolic and hyperbolic conformal type suggests there might be a rich theory of properly embedded minimal annuli in Nil_3 .

These “horizontal catenoids” are obtained using the Weierstrass-type representation for minimal surfaces in Nil_3 [Dan06]. We start with a suitable harmonic map into the hyperbolic disk; this harmonic map is expressed explicitly in terms of a solution of an ODE and it will be the Gauss map of the surface. We integrate the equations and then solve a period problem. Consequently we obtain an explicit expression for these horizontal catenoids in terms of a solution of an ODE (see proposition 5.1).

The second main point in this paper is to prove a half-space theorem. Discovered by Hoffman and Meeks [HM90], the half-space theorem for minimal surfaces of \mathbb{R}^3 is used to understand the global geometry of proper examples. In Nil_3 , we prove that our family of “horizontal catenoids” converges to a punctured vertical plane and then we obtain a “vertical half-space theorem” (vertical planes are defined in section 2).

Theorem (theorem 6.3). *Let Σ be a properly immersed minimal surface in Nil_3 . Assume that Σ is contained on one side of a vertical plane P . Then Σ is a vertical plane parallel to P .*

We next deal with complete graphs. There is a natural notion of graph in Nil_3 . Indeed, Nil_3 admits a Riemannian fibration $\pi : \text{Nil}_3 \rightarrow \mathbb{R}^2$ over the Euclidean plane. We will denote by ξ a unit vector field tangent to the fibers and we will call it a vertical vector field; it is a Killing field. Then a surface Σ in Nil_3 is a multigraph if it is transverse to ξ , it is a graph if it is transverse to ξ and $\pi|_\Sigma : \Sigma \rightarrow \mathbb{R}^2$ is injective, and it is an entire graph if it is transverse to ξ and $\pi|_\Sigma : \Sigma \rightarrow \mathbb{R}^2$ is bijective.

A natural problem is to determine if a complete multigraph is necessarily entire. We solve this problem using our half-space theorem and applying the arguments of [CR07] and [HRS07]. This is the following theorem.

Theorem (theorem 3.1). *Let Σ be a complete minimal immersed surface in Nil_3 . If Σ is transverse to the vertical Killing field ξ , then Σ is an entire graph.*

Also, recently, Fernandez and Mira solved the Bernstein problem in Nil_3 . We denote by \mathbb{C} the complex plane and by \mathbb{D} the unit disk $\{z \in \mathbb{C}; |z| < 1\}$.

Theorem ([FM07b]). *Let Q be a holomorphic quadratic differential on \mathbb{D} or a non-identically zero holomorphic quadratic differential on \mathbb{C} . Then there exists a 2-parameter family of generically non-congruent entire minimal graphs in Nil_3 whose Abresch-Rosenberg differential is Q .*

Conversely, all entire minimal graphs belong to these families.

Together with our theorem 3.1, this gives a classification of all complete minimal graphs in Nil_3 .

In this paper we will also deal with CMC $\frac{1}{2}$ surfaces in $\mathbb{H}^2 \times \mathbb{R}$. For these surfaces, Hauswirth, Rosenberg and Spruck [HRS07] proved a half-space type theorem and used it to show that complete multigraphs are entire, i.e., graphs over the whole hyperbolic plane \mathbb{H}^2 .

Our proof of theorem 6.3 is different from the proof of the half-space type theorem in [HRS07]: the main point in their proof is the construction of a continuous family

of compact annuli with boundaries, contained between two horocylinders of $\mathbb{H}^2 \times \mathbb{R}$, and converging to one of the horocylinders; they use Schauder's fixed point theorem in a quasi-linear equation; they have to control the mean curvature vector in the maximum principle. Our proof uses the family of complete annuli and the classical geometrical argument of Hoffman and Meeks [HM90].

Also, by our theorem 3.1 and [FM07b] we show that entire minimal graphs in Nil_3 correspond exactly to entire CMC $\frac{1}{2}$ graphs in $\mathbb{H}^2 \times \mathbb{R}$ by the sister surface correspondence (corollary 3.3). Hence we obtain a classification of all entire CMC $\frac{1}{2}$ graphs in $\mathbb{H}^2 \times \mathbb{R}$; this solves the Bernstein problem for CMC $\frac{1}{2}$ graphs in $\mathbb{H}^2 \times \mathbb{R}$. This is the following theorem.

Theorem. *Let Q be a holomorphic quadratic differential on \mathbb{D} or a non-identically zero holomorphic quadratic differential on \mathbb{C} . Then there exists a 2-parameter family of generically non-congruent entire CMC $\frac{1}{2}$ graphs in $\mathbb{H}^2 \times \mathbb{R}$ whose Abresch-Rosenberg differential is Q .*

Conversely, all entire CMC $\frac{1}{2}$ graphs belong to these families.

Observe that this theorem could not be obtained using the method of [FM07b]; indeed their solution of the Bernstein problem for minimal surfaces in Nil_3 is based on the relations between minimal immersions in Nil_3 and spacelike CMC immersions in Minkowski space \mathbb{L}^3 , and their arguments do not apply in our case.

We also construct a one-parameter family of properly embedded CMC $\frac{1}{2}$ annuli in $\mathbb{H}^2 \times \mathbb{R}$ which are analogous to our minimal horizontal catenoids in Nil_3 .

Theorem (see section 8). *There exists a one-parameter family $(\mathcal{C}_\alpha)_{\alpha>0}$ of properly embedded CMC $\frac{1}{2}$ annuli in $\mathbb{H}^2 \times \mathbb{R}$, called “horizontal catenoids”, having the following properties:*

- *the annulus \mathcal{C}_α is not invariant by a one-parameter group of isometries,*
- *the annulus \mathcal{C}_α is invariant by the reflections with respect to a horizontal plane and two orthogonal vertical planes, and it is a bigraph over some domain in a horizontal plane,*
- *the annulus \mathcal{C}_α is conformally equivalent to $\mathbb{C} \setminus \{0\}$.*

The curve of intersection of \mathcal{C}_α with its horizontal symmetry plane is similar to the profile curve of a rotational CMC 1 catenoid cousin in hyperbolic space \mathbb{H}^3 (see [Bry87], [UY93]). Moreover, this family converges to two punctured horocylinders tangent to each other. Hence it can give an alternative proof of the half-space type theorem of [HRS07].

These annuli are the sister surfaces of helicoidal type minimal surfaces in Nil_3 (see section 7). They are obtained in a way similar to that of the minimal horizontal catenoids in Nil_3 : we start from a suitable harmonic map into the hyperbolic disk and integrate the equations of [FM07b]; the period problem is solved automatically using the symmetries. Hence we obtain an explicit expression in terms of a solution of an ODE.

The paper is organized as follows. In section 2 we introduce material about harmonic maps, minimal surfaces in Nil_3 and CMC $\frac{1}{2}$ surfaces in $\mathbb{H}^2 \times \mathbb{R}$. In section 3 we give the proof and consequences of theorem 3.1 assuming the vertical half-space theorem. In section 4 we present the family of harmonic maps that will be used in the sequel. Section 5 is devoted to the construction of properly embedded minimal annuli in Nil_3 . In section 6 we prove our half-space theorem. In section 7, we construct periodic helicoidal surfaces with horizontal “axis”. In section 8 we

construct properly embedded CMC $\frac{1}{2}$ annuli in $\mathbb{H}^2 \times \mathbb{R}$. Finally, in section 9 we give the proofs of technical lemmas.

2. PRELIMINARIES

2.1. Harmonic maps and holomorphic quadratic differentials. In the following, we will use the unit disk model for \mathbb{H}^2 . We will note $\mathbb{H}^2 = (\mathbb{D}, \sigma^2(u)|du|^2)$ the disk with the hyperbolic metric $\sigma^2(u)|du|^2 = \frac{4}{(1-|u|^2)^2}|du|^2$. The harmonic map equation is

$$(1) \quad g_{z\bar{z}} + \frac{2\bar{g}}{(1-|g|^2)}g_z g_{\bar{z}} = 0.$$

In the theory of harmonic maps there is a global object to consider: the holomorphic quadratic Hopf differential associated to g ,

$$(2) \quad Q(g) = \phi(z)dz^2 = (\sigma \circ g)^2 g_z \bar{g}_z dz^2.$$

The function ϕ depends on the choice of the complex coordinate z , whereas $Q(g)$ does not. If $Q(g)$ is holomorphic then g is harmonic. We define the function $\omega = \frac{1}{2} \log \frac{|g_z|}{|g_{\bar{z}}|}$.

For a given holomorphic quadratic differential $Q = \phi(z)dz^2$, Wan [Wan92] on \mathbb{D} , Wan and Au [WA94] on \mathbb{C} , constructed a unique (up to isometries) harmonic map $g : \Sigma \rightarrow \mathbb{H}^2$ with non negative Jacobian and such that the metric

$$\tau|dz|^2 = 4(\sigma \circ g)^2 |g_z|^2 |dz|^2 = 4e^{2\omega} |\phi| |dz|^2$$

is complete. To do that, they construct a spacelike CMC $\frac{1}{2}$ in Minkowski space \mathbb{L}^3 with Gauss map g and metric $\tau|dz|^2$. First they solve the Gauss equation for the local theory of these surfaces:

$$(3) \quad \Delta_0 \omega = 2 \sinh(2\omega) |\phi|$$

where $\Delta_0 \omega = 4\omega_{z\bar{z}}$. The Codazzi equation is a consequence of the fact that ϕ is holomorphic. Then a maximum principle of Cheng and Yau [CY75] implies that there is a unique solution of (3) with complete metric $\tau|dz|^2$. Then by integration of the Gauss and Codazzi equations there is a unique (up to isometries) spacelike CMC $\frac{1}{2}$ immersion $\tilde{X} = (\tilde{F}, \tilde{h})$ in the Minkowski space \mathbb{L}^3 . The Gauss map of \tilde{X} is the map $g = \psi \circ \tilde{N} : \Sigma \rightarrow \mathbb{D}$, where ψ is the stereographic projection with respect to the southern pole of the quadric $\{|v|^2 = -1\}$. The data (Q, τ) determine g uniquely (up to isometries). When $\tau|dz|^2$ is complete we say that g is τ -complete.

In section 4, we will construct a family of harmonic maps with $Q = c dz^2$ ($c \in \mathbb{C}$) and not necessarily τ -complete. We will use these examples to construct our horizontal catenoids.

We describe a notion of conjugate harmonic map. It is known that a harmonic map g with Q having even zeroes induces a minimal surface in $\mathbb{H}^2 \times \mathbb{R}$. The immersion is given $X = (g, \operatorname{Re} \int -2i\sqrt{Q})$ and the induced metric is $ds^2 = 4 \cosh^2 \omega |Q|$ ([HR07]). Conversely, if $X = (g, t)$ is a conformal minimal immersion then g is harmonic and $Q(t) = -(t_z)^2 dz^2$ is a holomorphic quadratic differential with $Q(t) = Q(g)$.

Definition 2.1. *Two conformal minimal immersion $X, X^* : \Sigma \rightarrow \mathbb{H}^2 \times \mathbb{R}$ are conjugate if they induce the same metric on Σ and if we have $Q(g^*) = -Q(g)$.*

In [HSET05] and [Dan04], it is proven that the conjugate immersion exists. If $X^* = (g^*, h^*)$, then we say that g^* is the conjugate harmonic map of g . In particular we will use $Q(g^*) = -Q(g)$ and $\cosh \omega^* = \cosh \omega$ (and $\tau = \tau^*$).

2.2. Minimal surfaces in the Heisenberg group. In the sequel, we use the exponential coordinates to identify the Heisenberg group Nil_3 with $(\mathbb{R}^3, d\sigma^2)$, where $d\sigma^2$ given by

$$d\sigma^2 = dx_1^2 + dx_2^2 + \left(dx_3 + \frac{1}{2}(x_2 dx_1 - x_1 dx_2) \right)^2.$$

The projection $\pi : \text{Nil}_3 \rightarrow \mathbb{R}^2, (x_1, x_2, x_3) \mapsto (x_1, x_2)$ is a Riemannian fibration. We consider the left-invariant orthonormal frame (E_1, E_2, E_3) defined by

$$E_1 = \frac{\partial}{\partial x_1} - \frac{x_2}{2} \frac{\partial}{\partial x_3}, \quad E_2 = \frac{\partial}{\partial x_2} + \frac{x_1}{2} \frac{\partial}{\partial x_3}, \quad E_3 = \frac{\partial}{\partial x_3} = \xi.$$

A vector is said to be vertical if it is proportional to ξ and horizontal if it is orthogonal to ξ . A surface is a multigraph if ξ is nowhere tangent to it, i.e., if the restriction of π to the surface is a local diffeomorphism. The isometry group of Nil_3 is 4-dimensional and has two connected components: isometries preserving the orientation of the fibers and the base of the fibration, and those reversing both of them. Vertical translations are isometries. The Heisenberg group Nil_3 is a homogeneous manifold.

Lemma 2.2. *Let $X : \Sigma \rightarrow \text{Nil}_3$ be an immersion. Let N be the unit normal vector to X and let \tilde{N} be the Euclidean unit normal vector to X considered as an immersion into \mathbb{R}^3 . Then N points up if and only if \tilde{N} points up.*

Proof. We consider a conformal coordinate $z = u + iv$. In the frame (E_1, E_2, E_3) we have

$$X_u = \begin{bmatrix} x_{1u} \\ x_{2u} \\ x_{3u} + \frac{1}{2}(x_2 x_{1u} - x_1 x_{2u}) \end{bmatrix}, \quad X_v = \begin{bmatrix} x_{1v} \\ x_{2v} \\ x_{3v} + \frac{1}{2}(x_2 x_{1v} - x_1 x_{2v}) \end{bmatrix}.$$

Thus the third coordinate of $X_u \times X_v$ is $x_{1u}x_{2v} - x_{1v}x_{2u}$ which is also the third coordinate in the frame $\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$ of $X_u \wedge X_v$, where \wedge is the Euclidean vector product. \square

We will call vertical planes surfaces of equation $a_1 x_1 + a_2 x_2 = b$ for some constants a_1, a_2 and b with $(a_1, a_2) \neq (0, 0)$. Such surfaces are minimal and flat, but not totally geodesic. Two vertical planes will be said to be parallel if their images by the projection π are two parallel straight lines in \mathbb{R}^2 .

A graph $\{x_3 = f(x_1, x_2)\}$ is minimal if f satisfies the quasi-linear equation

$$(1 + q^2)r - 2pqs + (1 + p^2)t = 0$$

with

$$p = f_{x_1} + \frac{x_2}{2}, \quad q = f_{x_2} - \frac{x_1}{2}, \\ r = f_{x_1 x_1}, \quad s = f_{x_1 x_2}, \quad t = f_{x_2 x_2}.$$

The Bernstein problem deals with the existence and the unicity of entire solutions of this quasi-linear equation. We use conformal parametrization of surfaces. Let $X : \Sigma \rightarrow \text{Nil}_3$ be a conformal immersion. We denote by $F = \pi \circ X$ the horizontal projection of X and $h : \Sigma \rightarrow \mathbb{R}$ the third coordinate of X . We regard F as a

complex-valued function, identifying \mathbb{C} and \mathbb{R}^2 . We denote the metric by $ds^2 = \lambda|dz|^2$ and by $N : \Sigma \rightarrow \mathbb{S}^2$ the unit normal vector to X , where \mathbb{S}^2 is the unit sphere in the Lie algebra of Nil_3 .

The Gauss map of X is the map $g = \psi \circ N : \Sigma \rightarrow \bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, where ψ is the stereographic projection with respect to the southern pole, i.e., g is defined by

$$N = \frac{1}{1 + |g|^2} \begin{bmatrix} 2 \operatorname{Re} g \\ 2 \operatorname{Im} g \\ 1 - |g|^2 \end{bmatrix}$$

in (E_1, E_2, E_3) . The first author proved in [Dan06] that the Gauss map g satisfies

$$(4) \quad (1 - |g|^2)g_{z\bar{z}} + 2\bar{g}g_zg_{\bar{z}} = 0.$$

It is important to keep in mind that $|g| = 1$ exactly at points where the surface is not transverse to ξ .

If Σ is a multigraph, then, up to a change of orientations, g takes values in the unit disc \mathbb{D} . When \mathbb{D} is endowed with the hyperbolic metric $\frac{4}{(1-|z|^2)^2}|dz|^2$, g is a harmonic map from Σ to \mathbb{H}^2 . Conversely, we can recover a minimal immersion from a given harmonic map using the following theorem.

Theorem 2.3 ([Dan06]). *Let Σ be a simply-connected Riemann surface. Let $g : \Sigma \rightarrow \mathbb{H}^2$ be a harmonic map that is nowhere antiholomorphic. Let $z_0 \in \Sigma$, $F_0 \in \mathbb{C}$ and $h_0 \in \mathbb{R}$.*

Then there exists a unique conformal minimal immersion $X : \Sigma \rightarrow \text{Nil}_3$ such that g is the Gauss map of X and $X(z_0) = (F_0, h_0)$.

Moreover the immersion $X = (F, h)$ satisfies

$$F_z = -4i \frac{g_z}{(1 - |g|^2)^2}, \quad F_{\bar{z}} = -4i \frac{g^2 \bar{g}_{\bar{z}}}{(1 - |g|^2)^2},$$

$$h_z = 4i \frac{\bar{g}g_z}{(1 - |g|^2)^2} - \frac{i}{4}(\bar{F}F_z - F\bar{F}_{\bar{z}}).$$

The metric of the immersion is given by

$$ds^2 = 16 \frac{(1 + |g|^2)^2}{(1 - |g|^2)^4} |g_z|^2 |dz|^2.$$

The hypothesis “nowhere antiholomorphic” forces $\lambda|dz|^2$ to be a metric without branch points. The metrics $\lambda|dz|^2$ and $\tau|dz|^2$ are related by

$$\lambda = \frac{\tau}{\nu^2}$$

where

$$\nu = \frac{1 - |g|^2}{1 + |g|^2}$$

is the third coordinate of N . In the case of a multigraph we have $0 < |\nu| \leq 1$, and so, by the above relation between λ and τ , it is clear that the metric $\lambda|dz|^2$ is complete if $\tau|dz|^2$ is complete.

In section 4 we will use a family of harmonic maps to construct explicitly minimal annuli which will be the unions of two non-complete graphs.

It is worth mentioning some recent results of Fernandez and Mira.

Theorem 2.4 ([FM07b]). *Every τ -complete nowhere antiholomorphic harmonic map induces an entire minimal graph in Nil_3 . Conversely, every entire minimal graph in Nil_3 admits a τ -complete harmonic Gauss map g .*

This theorem proves that, starting from a holomorphic quadratic differential Q , there is a one-to-one canonical way to associate a two-parameter family of entire minimal graphs in Nil_3 [FM07b].

This is not enough to prove that complete multigraphs of Nil_3 are entire graphs and then coming from a τ -complete harmonic Gauss map. This fact will be the object of section 3. In other words, we will prove that, for a multigraph, if $\lambda|dz|^2$ is complete, then $\tau|dz|^2$ is also complete.

2.3. Constant mean curvature $\frac{1}{2}$ surfaces in $\mathbb{H}^2 \times \mathbb{R}$. Abresch and Rosenberg [AR04] constructed a holomorphic quadratic differential Q_0 associated to CMC $\frac{1}{2}$ surfaces in $\mathbb{H}^2 \times \mathbb{R}$; it generalizes the Hopf differential associated to constant mean curvature surfaces of \mathbb{R}^3 . When the surface is a graph, Fernandez and Mira [FM07a] constructed a harmonic “hyperbolic Gauss map” from the surface to \mathbb{H}^2 whose associated Hopf differential is $Q = -Q_0$. In addition, given a harmonic map g from a surface to \mathbb{H}^2 plus some additional data (described below) they construct CMC $\frac{1}{2}$ graphs on $\mathbb{H}^2 \times \mathbb{R}$ with this harmonic map as Gauss map.

Fernandez and Mira constructed CMC $\frac{1}{2}$ multigraph immersions $X^* = (F^*, h^*) : \Sigma \rightarrow \mathbb{H}^2 \times \mathbb{R}$ depending on the data (Q, τ) .

Theorem 2.5 ([FM07a]). *Let Σ be a simply connected Riemann surface and $g^* : \Sigma \rightarrow \mathbb{H}^2$ be a harmonic map admitting data $(-Q, \tau)$. Then for any $\theta_0 \in \mathbb{C}$ there exists a unique CMC $\frac{1}{2}$ immersion $X^* = (F^*, h^*) : \Sigma \rightarrow \mathbb{H}^2 \times \mathbb{R}$ satisfying*

- $\tau = \lambda\nu^2$, where λ is the conformal factor of the metric of X^* and ν is the vertical coordinate of the unit normal Gauss map,
- $h_z^*(z_0) = \theta_0$.

Moreover, with $G = \left(\frac{2g^*}{1-|g^*|^2}, \frac{1+|g^*|^2}{1-|g^*|^2} \right)$ we have

$$F^* = \frac{8\text{Re}(G_z(4\bar{Q}h_z^* + \tau h_z^{*\bar{z}}))}{\tau^2 - 16|Q|^2} + G\sqrt{\frac{\tau + 4|h_z^*|^2}{\tau}}$$

and $h^* : \Sigma \rightarrow \mathbb{R}$ is the unique (up to an additive constant) solution to the differential system below with $h_z^*(z_0) = \theta_0$:

$$\begin{cases} h_{zz}^* = (\log \tau)_z h_z^* + Q\sqrt{\frac{\tau + 4|h_z^*|^2}{\tau}}, \\ h_{z\bar{z}}^* = \frac{1}{4}\sqrt{\tau(\tau + 4|h_z^*|^2)}. \end{cases}$$

The metric can be expressed as

$$\lambda = \frac{\tau}{\nu^2} = \tau + 4|h_z^*|^2, \quad \nu = \sqrt{\frac{\tau}{\tau + 4|h_z^*|^2}}.$$

By the above relation between λ and τ , it is clear that the metric $ds^2 = \lambda|dz|^2$ is complete if $\tau|dz|^2$ is complete. Thus, associated to a holomorphic quadratic differential Q , one obtains a complete CMC $\frac{1}{2}$ multigraph in $\mathbb{H}^2 \times \mathbb{R}$.

It is known from [Dan07] that a CMC $\frac{1}{2}$ immersion $F^* = (X^*, h^*)$ is locally isometric to a minimal immersion $X = (F, h)$ in Nil_3 . These two immersions are called sister immersions. The third coordinate ν of the unit normal vector

of X and X^* remains unchanged by this correspondence. In particular the sister surface of a multigraph is a multigraph. The harmonic Gauss maps are conjugate ($Q(g) = -Q(g^*)$ and $\tau = \tau^*$).

We mention the following result of Fernandez and Mira :

Proposition 2.6 ([FM07b]). *If $X = (F, h)$ is a CMC $\frac{1}{2}$ minimal graph in $\mathbb{H}^2 \times \mathbb{R}$ with a τ -complete harmonic Gauss map g , then X is an entire graph.*

3. COMPLETE GRAPHS

In this section we use the half-space theorem 6.3 to obtain results on complete graphs in Nil_3 and $\mathbb{H}^2 \times \mathbb{R}$.

Theorem 3.1. *Let Σ be a complete minimal surface in Nil_3 . If Σ is transverse to the vertical Killing field ξ , then Σ is an entire graph.*

Corollary 3.2. *Let Σ be a complete minimal surface in Nil_3 . If Σ is transverse to the vertical Killing field ξ , then its Gauss map is τ -complete.*

Proof. From [FM07b] we know that an entire graph has a τ -complete Gauss map. \square

Corollary 3.3. *A minimal surface in Nil_3 is an entire graph if and only if its CMC $\frac{1}{2}$ sister surface in $\mathbb{H}^2 \times \mathbb{R}$ is an entire graph.*

Proof. By [FM07b], an entire graph of Nil_3 has a τ -complete Gauss map. Then again by [FM07b] the sister CMC $\frac{1}{2}$ surface is entire in $\mathbb{H}^2 \times \mathbb{R}$ (this fact comes from the completeness of $\tau|dz|^2$).

Conversely, the sister of an entire CMC $\frac{1}{2}$ graph in $\mathbb{H}^2 \times \mathbb{R}$ is a complete multigraph and then entire in Nil_3 by our theorem 3.1. \square

Corollary 3.4. *Let Σ be a complete CMC $\frac{1}{2}$ surface in $\mathbb{H}^2 \times \mathbb{R}$. If Σ is a multigraph, then its Gauss map is τ -complete.*

Proof. By the theorem of Hauswirth, Rosenberg and Spruck [HRS07], Σ is an entire graph hence its sister also (Corollary 3.3). From [FM07b] we know that an entire graph is τ -complete. \square

We now prove theorem 3.1. The proof is an adaptation to our case of the proof of Theorem 1.2 in [HRS07]. However, we give a detailed proof for the reader's convenience and since we need to take care about the meaning of horizontal vectors in Nil_3 . Lemma 3.5 below is inspired by the work of Collin and Rosenberg [CR07].

Let Σ be a complete minimal surface in Nil_3 such that Σ is transverse to the vertical Killing field ξ .

We assume that Σ is not entire. We denote by N the unit normal vector field to Σ . Since Σ is a multigraph, it is orientable. The function $\nu = \langle N, \xi \rangle$ is a non-vanishing Jacobi function on Σ , so Σ is strongly stable and thus has bounded curvature. Hence there is $\delta > 0$ such that, for each $p \in \Sigma$, there is a piece $G(p)$ of Σ around p that is a graph (in exponential coordinates) over the disk $D_{2\delta}(p) \subset T_p\Sigma$ of radius 2δ centered at the origin of $T_p\Sigma$. This graph $G(p)$ has bounded geometry. The δ is independant of p and the bound on the geometry of $G(p)$ is uniform as well.

We denote by $F(p)$ the image of $G(p)$ by the vertical translation mapping p at height $x_3 = 0$, and by $F_O(p)$ the image of $G(p)$ by the translation mapping p to $O = (0, 0, 0) \in \text{Nil}_3$.

In the sequel, we will call x_3 -graphs graphs with respect to the Riemannian fibration $\pi : \text{Nil}_3 \rightarrow \mathbb{R}^2$ (as explained in section 2).

We will identify vectors at different points of Nil_3 by left multiplication, and horizontal vectors with vectors in \mathbb{R}^2 .

Lemma 3.5. *Let (p_n) be a sequence of points of Σ such that $N(p_n)$ has a horizontal limit N_∞ when $n \rightarrow +\infty$. Then there is a subsequence of $(F_O(p_n))_{n \in \mathbb{N}}$ that converges to a δ -piece P_δ around O of the vertical plane P passing through O and having N_∞ as unit normal vector at O .*

The convergence is in the \mathcal{C}^2 -topology. By δ -piece around O we mean a piece of P containing all the points $p \in P$ such that $x_3(p) \in [-\delta, \delta]$ and $\pi(p)$ belongs to the closed segment of $\pi(P) \subset \mathbb{R}^2$ centered at $\pi(O)$ and of length 2δ .

Proof. Let P be the vertical plane passing through O and having N_∞ as unit normal vector at O . We endow P with the orientation induced by N_∞ .

Since the $F_O(p_n)$ have bounded geometry and are graphs over $D_{2\delta}(p_n) \subset T_O(F_O(p_n))$, the $F_O(p_n)$ are bounded exponential graphs over a δ -piece $P_\delta \subset P$ around O . Thus a subsequence of these graphs converges to a piece of a minimal surface F_∞ , which is tangent to P_δ at O and which is an exponential graph over P_δ . It suffices to show that F_∞ is a piece of P .

If this is not the case, then by Theorem 5.3 in [CM99], in the neighbourhood of O , the intersection of F_∞ and the vertical plane consists of m ($m \geq 2$) curves meeting at O . These curves separate F_∞ into $2m$ connected components and adjacent components lie on opposite sides of the vertical plane. Hence in a neighborhood of O , the Euclidean unit normal vector to F_∞ alternates from pointing up to pointing down as one goes from one component to the other. This is also the case for $F_O(p_n)$, for n large, since $F_O(p_n)$ converges to F_∞ in \mathcal{C}^2 -topology. Then by lemma 2.2, the unit normal vector to $F_O(p_n)$ for the metric of Nil_3 also alternates from pointing up to down. This contradicts the fact that $F_O(p_n)$ is transverse to ξ . \square

We now consider a piece of Σ that is the x_3 -graph of a function f defined on the open disk B_R of radius R centered at some point A of \mathbb{R}^2 . Since Σ is not an entire graph, we choose the largest R such that f exists.

In the sequel, for any point $q \in B_R$ we will write $F(q)$, $N(q)$, etc. instead of $F(q, f(q))$, $N(q, f(q))$, etc.

Let $q \in \partial B_R$ be such that f does not extend to any neighbourhood of q (to a function satisfying the minimal graph equation).

Lemma 3.6. *There exists a unit horizontal vector $N_\infty(q)$ such that, for any sequence $q_n \in B_R$ converging to q , $N(q_n) \rightarrow N_\infty(q)$ when $n \rightarrow +\infty$. Moreover, $N_\infty(q)$ is normal to ∂B_R at q .*

Proof. We first observe that $\nu(q_n) \rightarrow 0$ when $n \rightarrow +\infty$ (i.e. tangent planes become vertical); otherwise, the exponential graph of bounded geometry $G(q_n)$ would extend to an x_3 -graph beyond q for q_n close enough to q , and thus the map f would extend, which is a contradiction.

Let $N_\infty(q)$ be the horizontal unit vector at q normal to ∂B_R and pointing inside B_R . We will now prove that $N(q_n) \rightarrow N_\infty(q)$.

Assume that there exists a subsequence such that $N(q_n)$ converges to a horizontal vector $v \neq N_\infty(q)$. By lemma 3.5 there exists a subsequence such that the pieces $F(q_n)$ converge to a δ -piece of a vertical plane Q having v as unit normal vector at q . Since $N_\infty(q)$ is normal to ∂B_R at q , there are points of $\pi(Q_\delta)$ in B_R . Consequently there is a point $\hat{q} \in \pi(Q_\delta) \cap B_R$ and a sequence (\hat{q}_n) of points of B_R converging to \hat{q} such that $(\hat{q}_n, f(\hat{q}_n)) \in G(q_n)$. Since the $F(q_n)$ converge to a δ -piece of a vertical plane in the \mathcal{C}^2 -topology, Σ has a horizontal normal at $(\hat{q}, f(\hat{q}))$, which contradicts the fact that Σ is transverse to ξ . \square

We denote by P the vertical plane passing through q and having $N_\infty(q)$ as unit normal vector at q . Without loss of generality we can assume that $q = (R, 0) \in \mathbb{R}^2$ and that P is the vertical plane of equation $x_1 = R$ in Nil_3 . We will say that a point in Nil_3 is on the left side (respectively, on the right side) of P if $x_1 < R$ (respectively, $x_1 > R$).

Lemma 3.7. *We have $f(t, 0) \rightarrow \pm\infty$ when $t \rightarrow R$.*

Proof. Let $\varphi(t) = f(t, 0)$ and $\gamma(t) = (t, 0, \varphi(t))$.

We first claim that for t close enough to R we have $\varphi'(t) \neq 0$. Indeed, assume that there exists t_0 such that $\varphi'(t_0) = 0$. We have $\gamma'(t) = E_1 + \varphi'(t)\xi$ so $\gamma'(t_0)$ is horizontal. Also, $G(\gamma(t_0))$ is an exponential graph over $D_{2\delta}(\gamma(t_0)) \subset T_{\gamma(t_0)}\Sigma$ and $T_{\gamma(t_0)}\Sigma$ contains the horizontal vector $\gamma'(t_0)$. Consequently, the projection of $G(\gamma(t_0))$ on \mathbb{R}^2 contains an open neighbourhood of $\{(t, 0); t_0 - \delta < t < t_0 + \delta\}$. Hence, if t_0 is close enough to R , this implies that f extends beyond $q = (R, 0)$, which is a contradiction. This proves the claim.

Thus we can assume that $\varphi(t)$ is increasing as t converges to R . If $\varphi(t)$ were bounded from above, then it would have a finite limit l and the curve $t \mapsto (t, 0, f(t, 0))$ in Σ would have finite length up till (q, l) . Since Σ is complete we would have $(q, l) \in \Sigma$, but then Σ would have a vertical tangent plane at (q, l) (otherwise f would extend to some neighbourhood of q), which gives a contradiction. \square

From now on we assume that $f(t, 0) \rightarrow +\infty$ when $t \rightarrow R$ (the case where $f(t, 0) \rightarrow -\infty$ is similar). We set

$$\Gamma = \pi(P) = \{(R, s); s \in \mathbb{R}\}$$

and, for $\varepsilon > 0$ and $s \in \mathbb{R}$,

$$\begin{aligned} U_\varepsilon &=]R - \varepsilon, R[\times \mathbb{R}, \\ \gamma_{s, \varepsilon} &= \{(x_1, s); R - \varepsilon < x_1 < R + \varepsilon\}, \\ \gamma_{s, \varepsilon}^+ &= \{(x_1, s); R - \varepsilon < x_1 < R\}. \end{aligned}$$

We fix $\varepsilon_0 > 0$ and consider a sequence (t_n) of real numbers such that $q_n = (t_n, 0)$ is in B_R , $q_n \rightarrow q$ when $n \rightarrow +\infty$ and such that

$$G = \bigcup_{n \in \mathbb{N}} G(q_n)$$

is connected. By lemmas 3.5 and 3.7 and the fact that the $G(q_n)$ are pieces of bounded geometry, G is asymptotic to a part of P as one goes up. Moreover we can choose the q_n close enough to R and to each other such that, for all $s \in [-\delta, \delta]$, the curve

$$C_s = \pi^{-1}(\gamma_{s, \varepsilon_0}) \cap G$$

is connected and has no horizontal or vertical tangents. This is possible since the $F(q_n)$ are \mathcal{C}^2 -close to P_δ and since Σ is transverse to ξ .

Lemma 3.8. *Each $G(q_n)$ is disjoint from P , and, for $s \in [-\delta, \delta]$, C_s is an x_3 -graph over $\gamma_{s, \varepsilon(s)}^+$ for some $\varepsilon(s) \in (0, \varepsilon_0]$. Moreover, $\varepsilon(s)$ can be chosen continuous.*

Proof. The curve C_s is an x_3 -graph over an interval in $\gamma_{s, \varepsilon_0}$. We show that this interval is in $\gamma_{s, \varepsilon_0}^+$.

Suppose this is not the case for some $s_0 \in [-\delta, \delta]$. Then C_{s_0} has some points on the right side of P . But the curve C_0 stays on the left side of P (otherwise f would extend beyond q). So, for some $s_1 \in]0, s_0]$, C_{s_1} has points on both sides of P .

But G is asymptotic to a part of P as one goes up, so the curve C_{s_1} is asymptotic to $\pi^{-1}(R, s_1)$ as the height goes to $+\infty$. This obliges C_{s_1} to have a vertical tangent on the right side of P , which is a contradiction since Σ is transverse to ξ . \square

Consequently $\cup_{s \in [-\delta, \delta]} C_s$ is the x_3 -graph of a function g on $\cup_{s \in [-\delta, \delta]} \gamma_{s, \varepsilon(s)}^+$. The functions f and g coincide on the intersection of their domains of definition. The graph of g on each $\gamma_{s, \varepsilon(s)}^+$ is the curve C_s and the graph of g is asymptotic to P as the height goes to $+\infty$.

We can apply this process again replacing C_0 by the curve C_δ , then $C_{-\delta}$, and so on. Analytic continuation yields an extension h of g to a domain Ω contained on the left side of Γ . The domain Ω is an open neighbourhood of Γ in its left side. We have $h \rightarrow +\infty$ as one approaches Γ in Ω ; the graph of h is asymptotic to P as the height goes to $+\infty$.

Lemma 3.9. *There exists $\varepsilon > 0$ such that Ω contains U_ε .*

Proof. The surface Σ contains a graph over Ω composed of curves C_s such that each C_s is a graph over $\gamma_{s, \varepsilon(s)}^+$ for some $\varepsilon(s) > 0$. Also, for each $s \in \mathbb{R}$, $h(t, s)$ is strictly increasing in t when $t \rightarrow R$ and $h(t, s) \rightarrow +\infty$ when $t \rightarrow R$.

Let $\varepsilon_1 \in (0, \delta)$ such that $\varepsilon_1 \leq \varepsilon(s)$ for all $s \in [-\delta, \delta]$.

Suppose that for some $s_0 \in \mathbb{R}$ we have $\varepsilon(s_0) < \varepsilon_1$. We set $\varphi(t) = h(t, s_0)$ and consider the curve

$$c : (R - \varepsilon(s_0), R) \ni t \mapsto (t, s_0, \varphi(t)) \in \Sigma.$$

We first claim that this curve has no horizontal tangent; indeed, if there is some t_0 such that $c'(t_0)$ is horizontal, then since $G(c(t_0))$ is an exponential graph over $D_{2\delta}(c(t_0)) \subset T_{c(t_0)}\Sigma$, and since $t_0 + \delta > R$, then the curve c would go on the right side of P , which is a contradiction.

We have $c'(t) = E_1 + (\varphi'(t) + \frac{s_0}{2})\xi$, so the fact that c has no horizontal tangent implies that $\varphi'(t) > -\frac{s_0}{2}$. This implies that $\varphi(t)$ cannot tend to $+\infty$ when $t \rightarrow R - \varepsilon(s_0)$. So $\varphi(t) \rightarrow -\infty$ when $t \rightarrow R - \varepsilon(s_0)$ (otherwise the curve c would extend or have a vertical tangent at $R - \varepsilon(s_0)$).

The previous discussion where we showed that the graph over Ω exists and is asymptotic to P now applies to show that there is a vertical plane \tilde{P} passing through $\tilde{p} = (R - \varepsilon(s_0), s_0, 0) \in \text{Nil}_3$ such that a δ -neighbourhood of c in Σ is asymptotic to a δ -vertical strip in \tilde{P} as one goes down to $-\infty$. We know this δ -neighbourhood of c in Σ is asymptotic to a δ -vertical strip in P as one goes up to $+\infty$.

For each $s \in [s_0 - \delta, s_0 + \delta]$, the curve $c_s : t \mapsto (t, s, h(t, s))$ is asymptotic to some vertical line in \tilde{P} as one goes down to $-\infty$. By analytic continuation of the δ -neighbourhoods, one continues this process along Γ .

If P and \tilde{P} are parallel, then the process continues along all of Γ and Ω is the region bounded by $\pi(P)$ and $\pi(\tilde{P})$. Then all the $\varepsilon(s)$ are equal, and this concludes the proof.

So we can assume that P and \tilde{P} intersect along some vertical line $\pi^{-1}(\hat{p})$. Let us write $\hat{p} = (s_1, R)$. Consider the curves c_s as s goes from s_0 to s_1 ; they are graphs that become vertical both when the height goes to $+\infty$ and $-\infty$. Let $p(s)$ be the point of C_s at height 0; then, when $s \rightarrow s_1$, the path $p(s)$ has finite length (since the geometry of Σ is bounded), so, since Σ is complete, $p(s)$ converges to a point of Σ , and the tangent plane at this point is vertical. This contradicts the fact that Σ is transverse to ξ . \square

We can now complete the proof of theorem 3.1.

Proof of theorem 3.1. We showed that Σ contains a graph G over some U_ε which is asymptotic to P as one approaches Γ in U_ε . We apply the proof of the vertical half-space theorem 6.3 (this theorem is stated for complete surfaces without boundary, but the proof still works in our case since G is proper in some tubular neighbourhood of P despite it has a non-compact boundary). This shows that such a graph G cannot exist.

Consequently, Σ is entire, and so it is an entire graph. \square

4. THE FAMILY OF HARMONIC MAPS

In this section we construct a family of harmonic maps that we will use to construct annuli. This family is derived from the two-parameter family of minimal surfaces of $\mathbb{H}^2 \times \mathbb{R}$ constructed in [Hau06].

For $\alpha > 0$ and $\theta \in \mathbb{R}$ we define $g : \mathbb{C} \rightarrow \bar{\mathbb{C}}$ by

$$g(u + iv) = \frac{\sin \varphi(u) + i \sinh(\alpha v + \beta(u))}{\cos \varphi(u) + \cosh(\alpha v + \beta(u))} = \frac{\cosh(\alpha v + \beta(u)) - \cos \varphi(u)}{\sin \varphi(u) - i \sinh(\alpha v + \beta(u))}$$

where φ satisfies the following ODE:

$$(5) \quad \varphi'^2 = \alpha^2 + \cos(2\theta) \cos^2 \varphi - \frac{\sin^2(2\theta)}{4\alpha^2} \cos^4 \varphi,$$

and where β is defined by

$$\beta' = \frac{\sin(2\theta)}{2\alpha} \cos^2 \varphi, \quad \beta(0) = 0.$$

The function φ is defined on the whole \mathbb{R} . We will study this function φ in lemma 4.2. We also set

$$A = \alpha v + \beta(u), \quad D = \cos \varphi + \cosh A.$$

We notice that

$$(6) \quad 1 - |g|^2 = \frac{2 \cos \varphi}{D}.$$

Proposition 4.1. *The function g satisfies*

$$(1 - |g|^2)g_{z\bar{z}} + 2\bar{g}g_zg_{\bar{z}} = 0$$

and its Hopf differential is

$$Q = \frac{1}{4}e^{-2i\theta}dz^2.$$

Proof. To see that g satisfies the equation, it suffices to see that

$$Q = \frac{4}{(1 - |g|^2)^2} g_z g_{\bar{z}} dz^2$$

is holomorphic.

We compute

$$g_u = \frac{\varphi' + i\beta'}{D^2} (1 + \cos \varphi \cosh A + i \sin \varphi \sinh A),$$

$$g_v = \frac{i\alpha}{D^2} (1 + \cos \varphi \cosh A + i \sin \varphi \sinh A).$$

From this and (6) we get

$$Q = \frac{\varphi'^2 - (\alpha + i\beta')^2}{4 \cos^2 \varphi} dz^2.$$

Using (5) and the definition of β we get $Q = \frac{1}{4} e^{-2i\theta} dz^2$. \square

For $\alpha > 0$ and $\theta \in \mathbb{R}$, we set

$$C = C_{\alpha, \theta} = \frac{\sin(2\theta)}{2\alpha}, \quad P_{\alpha, \theta}(x) = \alpha^2 + \cos(2\theta)x^2 - C_{\alpha, \theta}^2 x^4,$$

so that (5) is equivalent to

$$\varphi'^2 = P_{\alpha, \theta}(\cos \varphi).$$

We set $\theta_\alpha^+ = \frac{\pi}{2}$ if $\alpha > 1$ and $\theta_\alpha^+ = \frac{1}{2} \arccos(1 - 2\alpha^2) \in (0, \frac{\pi}{2}]$ if $\alpha \leq 1$. Let $\Omega = \{(\alpha, \theta) \in \mathbb{R}^2; \alpha > 0, \theta \in (-\theta_\alpha^+, \theta_\alpha^+)\}$.

If $2\theta \notin \pi\mathbb{Z}$, we have

$$P_{\alpha, \theta}(x) = C_{\alpha, \theta}^2 (\rho_{\alpha, \theta}^- - x^2)(\rho_{\alpha, \theta}^+ + x^2)$$

with

$$\rho_{\alpha, \theta}^- = \frac{2\alpha^2}{1 - \cos(2\theta)}, \quad \rho_{\alpha, \theta}^+ = \frac{2\alpha^2}{1 + \cos(2\theta)}.$$

Thus, if $2\theta \notin \pi\mathbb{Z}$ and $(\alpha, \theta) \in \Omega$, then $\rho_{\alpha, \theta}^- > 1$. Also, we have $P_{\alpha, 0}(x) = \alpha^2 + x^2$. From this we deduce that

$$\forall (\alpha, \theta) \in \Omega, \forall x \in [-1, 1], P_{\alpha, \theta}(x) > 0.$$

Thus, if $(\alpha, \theta) \in \Omega$, then the right term in (5) does not vanish.

Lemma 4.2. *Let $(\alpha, \theta) \in \Omega$. Let φ be the solution of (5) such that $\varphi(0) = 0$ and $\varphi'(0) \leq 0$. Then*

1. $\forall u, \varphi'(u) < 0$,
2. the function φ is a decreasing bijection from \mathbb{R} onto \mathbb{R} ,
3. there exists a real number $U > 0$ such that

$$\forall u \in \mathbb{R}, \varphi(u + U) = \varphi(u) - \pi,$$

4. the function φ is odd.

Proof. 1. Since the right term in (5) does not vanish, φ' does not vanish.

2. The right term in (5) is bounded by two positive constants c_1 and c_2 ; hence $-\sqrt{c_2} \leq \varphi' \leq -\sqrt{c_1}$, which proves that φ is defined on the entire \mathbb{R} and that $\varphi(u) \rightarrow -\infty$ when $u \rightarrow +\infty$ and $\varphi(u) \rightarrow +\infty$ when $u \rightarrow -\infty$.

3. There exists $U > 0$ such that $\varphi(U) = -\pi$. Then the function $\tilde{\varphi} : u \mapsto \varphi(u + U) + \pi$ satisfies (5) with $\tilde{\varphi}(0) = 0$ and $\tilde{\varphi}' < 0$; hence $\tilde{\varphi} = \varphi$.

4. The function $\hat{\varphi} : u \mapsto -\varphi(-u)$ satisfies (5) with $\hat{\varphi}(0) = 0$ and $\hat{\varphi}' < 0$; hence $\hat{\varphi} = \varphi$.

□

In the sequel we will use the function $G : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$G' = \frac{C^2 \cos^2 \varphi - \cos(2\theta)}{\alpha - \varphi'}, \quad G(0) = 0.$$

(We recall that $\alpha - \varphi' > 0$.) The functions β and G are odd and satisfy

$$(7) \quad \forall u \in \mathbb{R}, \beta(u + U) = \beta(u) + \beta(U), \quad \forall u \in \mathbb{R}, G(u + U) = G(u) + G(U).$$

Lemma 4.3. *We have*

$$(8) \quad \varphi\left(\frac{U}{2}\right) = -\frac{\pi}{2}, \quad \beta\left(\frac{U}{2}\right) = \frac{\beta(U)}{2}, \quad G\left(\frac{U}{2}\right) = \frac{G(U)}{2}.$$

Proof. We have $\varphi\left(\frac{U}{2}\right) = \varphi\left(-\frac{U}{2}\right) - \pi = -\varphi\left(\frac{U}{2}\right) - \pi$, which gives the first formula. We prove the other formulas in the same way. □

Lemma 4.4. *The following identities hold.*

$$(9) \quad \varphi' + \alpha = G' \cos^2 \varphi,$$

$$(10) \quad \varphi'' = -(\cos(2\theta) - 2C^2 \cos^2 \varphi) \sin \varphi \cos \varphi,$$

$$(11) \quad G'' \cos \varphi = (2\varphi' G' - \cos(2\theta) + 2C^2 \cos^2 \varphi) \sin \varphi,$$

$$(12) \quad G'' = \frac{2C^2 \alpha - \cos(2\theta) G'}{\alpha - \varphi'} \sin \varphi \cos \varphi,$$

$$(13) \quad G'' = (C^2 + G'^2) \sin \varphi \cos \varphi.$$

Proof. Formulas (9), (10), (11) and (12) are straightforward. Using (12) and the definition of G we get

$$(\varphi' - \alpha)^2 G'' = (\cos^2(2\theta) - C^2 \cos(2\theta) \cos^2 \varphi - 2\alpha C^2 (\varphi' - \alpha)) \sin \varphi \cos \varphi.$$

On the other hand we have

$$(\varphi' - \alpha)^2 (C^2 + G'^2) = \cos^2(2\theta) - C^2 \cos(2\theta) \cos^2 \varphi - 2\alpha C^2 (\varphi' - \alpha).$$

This proves (13). □

For $(\alpha, \theta) \in \Omega$ we set

$$L(\alpha, \theta) = \int_{-1}^1 \frac{2\alpha C_{\alpha, \theta}^2 x^2 - \alpha \cos(2\theta) + C_{\alpha, \theta}^2 x^2 \sqrt{P_{\alpha, \theta}(x)}}{\sqrt{(1-x^2)P_{\alpha, \theta}(x)}(\alpha + \sqrt{P_{\alpha, \theta}(x)})} dx.$$

We will prove in section 9 the following technical lemmas.

Lemma 4.5. *Let $\alpha > 0$. Then there exists a unique $\tilde{\theta}_\alpha \in (0, \theta_\alpha^+) \cap (0, \frac{\pi}{4})$ such that*

$$L(\alpha, \tilde{\theta}_\alpha) = 0.$$

Lemma 4.6. *We have*

$$\lim_{\alpha \rightarrow +\infty} \tilde{\theta}_\alpha = \frac{\pi}{4}.$$

5. HORIZONTAL CATENOIDS IN Nil_3

In this section we construct a one-parameter family of properly embedded minimal annuli in Nil_3 . We use the notations of section 4.

We will start from the map g which satisfies (1) (by proposition 4.1) outside points where $|g| = 1$ but which does not take values in \mathbb{D} . However, in this case we can still recover a minimal immersion (but not a multigraph) by theorem 2.3 provided the map we obtain is well-defined when $|g| = 1$ and provided the metric we obtain has no singularity. In fact these two kinds of problems do not appear in our case, as shown by the following proposition.

Proposition 5.1. *The conformal minimal immersion $X = (F, h) : \mathbb{C} \rightarrow \text{Nil}_3$ whose Gauss map is g is given (up to a translation) by*

$$\begin{aligned} F(u + iv) &= \frac{G'}{\alpha} \cos \varphi \sinh A - \frac{C}{\alpha} \sin \varphi \cosh A + i(Cv - G), \\ h(u + iv) &= -\frac{1}{\alpha} \left(G' \sin \varphi + \frac{C^2}{\alpha} \sin \varphi + \frac{(Cv - G)G'}{2} \cos \varphi \right) \sinh A \\ &\quad + \frac{1}{\alpha} \left(-C \cos \varphi + \frac{CG'}{\alpha} \cos \varphi + \frac{C(Cv - G)}{2} \sin \varphi \right) \cosh A. \end{aligned}$$

The metric of X is given by

$$ds^2 = (G'^2 + C^2) \cosh^2 A |dz|^2.$$

Proof. We first recover F using theorem 2.3 and the above computations. We get

$$\begin{aligned} F_z &= -\frac{i}{2 \cos^2 \varphi} (\varphi' + i\beta' + \alpha)(1 + \cos \varphi \cosh A + i \sin \varphi \sinh A), \\ F_{\bar{z}} &= -\frac{i}{2 \cos^2 \varphi} (\varphi' - i\beta' + \alpha)(1 - \cos \varphi \cosh A + i \sin \varphi \sinh A), \end{aligned}$$

hence

$$\begin{aligned} F_u &= \frac{\beta' \cos \varphi \cosh A - i(\varphi' + \alpha)(1 + i \sin \varphi \sinh A)}{\cos^2 \varphi} \\ &= C \cos \varphi \cosh A - iG'(1 + i \sin \varphi \sinh A), \\ F_v &= \frac{(\varphi' + \alpha) \cos \varphi \cosh A + i\beta'(1 + i \sin \varphi \sinh A)}{\cos^2 \varphi} \\ &= G' \cos \varphi \cosh A + iC(1 + i \sin \varphi \sinh A). \end{aligned}$$

This gives F .

Then we get

$$\begin{aligned} h_z &= \frac{G' + iC}{4} (2 \cos \varphi \sinh A + 2i \sin \varphi \cosh A \\ &\quad - \frac{G'}{\alpha} \cos \varphi \sinh A + \frac{C}{\alpha} \sin \varphi \cosh A \\ &\quad + i(Cv - G) \cos \varphi \cosh A - (Cv - G) \sin \varphi \sinh A). \end{aligned}$$

This gives h .

Using (6) and computations done in the proof of proposition 4.1 we get

$$1 + |g|^2 = \frac{2 \cosh A}{D}, \quad |g_z|^2 = \frac{(\varphi' + \alpha)^2 + \beta'^2}{4D^2},$$

and so by theorem 2.3 we obtain the formula. \square

Proposition 5.2. *Let $\alpha > 0$ and $\theta = \tilde{\theta}_\alpha$. Then the corresponding immersion X is simply periodic, i.e., there exists $Z \in \mathbb{C} \setminus \{0\}$ such that*

$$\forall z \in \mathbb{C}, X(z + Z) = X(z).$$

Proof. Let $C_\alpha = C_{\alpha, \tilde{\theta}_\alpha}$ and $P_\alpha(x) = P_{\alpha, \tilde{\theta}_\alpha}(x)$. We set

$$V = -\frac{\beta(U)}{\alpha}.$$

Then, by (7), for all $(u, v) \in \mathbb{R}^2$, we have $A(u + U + i(v + V)) = A(u + iv)$.

We claim that, for all $(u, v) \in \mathbb{R}^2$, we have $\text{Im } F(u + U + i(v + V)) = \text{Im } F(u + iv)$, i.e., that

$$(14) \quad \alpha G(U) + C\beta(U) = 0.$$

We have

$$G(U) = \int_0^U G'(u) du, \quad \beta(U) = \int_0^U \beta'(u) du.$$

We do the change of variables $x = \cos \varphi(u)$, hence $dx = -\varphi' \sin \varphi du = \varphi' \sqrt{1 - x^2} du$ since $\varphi \in [-\pi, 0]$. We get

$$G(U) = \int_{-1}^1 \frac{C_\alpha^2 x^2 - \cos(2\tilde{\theta}_\alpha)}{\sqrt{(1 - x^2)P_\alpha(x)}(\alpha + \sqrt{P_\alpha(x)})} dx,$$

$$\beta(U) = \int_{-1}^1 \frac{C_\alpha x^2}{\sqrt{(1 - x^2)P_\alpha(x)}} dx,$$

and so $\alpha G(U) + C\beta(U) = L(\alpha, \tilde{\theta}_\alpha) = 0$ by lemma 4.5. This proves the claim.

Hence $A(u + iv)$ and $\text{Im } F(u + iv) = Cv - G(u)$ are $(U + iV)$ -periodic. We set $Z = 2(U + iV)$ (we have $Z \neq 0$ since $U > 0$). Then it follows from the expressions of F and h that they are Z -periodic. \square

Definition 5.3. Let $\alpha > 0$. The surface given by X when $\theta = \tilde{\theta}_\alpha$ is called a horizontal catenoid of parameter α with respect to the x_2 -axis. It will be denoted \mathcal{C}_α .

The coordinates (x_1, x_2, x_3) of \mathcal{C}_α are

$$\begin{aligned} x_1 &= \frac{G'(u)}{\alpha} \cos \varphi(u) \sinh A - \frac{C}{\alpha} \sin \varphi(u) \cosh A, \\ x_2 &= \frac{C}{\alpha} A - \frac{C}{\alpha} \beta(u) - G(u), \\ x_3 &= -\frac{x_1 x_2}{2} + \frac{C}{\alpha} \left(\frac{G'(u)}{\alpha} - 1 \right) \cos \varphi(u) \cosh A \\ &\quad - \frac{1}{\alpha} \left(\frac{C^2}{\alpha} + G'(u) \right) \sin \varphi(u) \sinh A. \end{aligned}$$

We now study the geometry of \mathcal{C}_α . We first notice that

$$\begin{cases} x_1(u + U, v + V) &= -x_1(u, v), \\ x_2(u + U, v + V) &= x_2(u, v), \\ x_3(u + U, v + V) &= -x_3(u, v), \end{cases}$$

so \mathcal{C}_α is invariant by the rotation of angle π around the x_2 -axis. We also have

$$\begin{cases} x_1(-u, -v) &= -x_1(u, v), \\ x_2(-u, -v) &= -x_2(u, v), \\ x_3(-u, -v) &= x_3(u, v), \end{cases}$$

so \mathcal{C}_α is invariant by the rotation of angle π around the x_3 -axis. Since the composition of the rotations of angle π around the x_2 and x_3 axes is the rotation of angle π around the x_1 -axis, \mathcal{C}_α is also invariant by this rotation.

It will be convenient to use the following coordinates in Nil_3 :

$$(15) \quad y_1 = x_1, \quad y_2 = x_2, \quad y_3 = x_3 + \frac{x_1 x_2}{2}.$$

In these coordinates the metric of Nil_3 is given by

$$dy_1^2 + dy_2^2 + (dy_3 - y_1 dy_2)^2.$$

In particular, in a vertical plane of equation $y_2 = c$ ($c \in \mathbb{R}$), the pair (y_1, y_3) is a pair of Euclidean coordinates.

We now study the intersection of \mathcal{C}_α with a vertical plane of equation $y_2 = c$ ($c \in \mathbb{R}$). On \mathcal{C}_α , this intersection is given by

$$(16) \quad A = \frac{\alpha}{C}c + \beta(u) + \frac{\alpha}{C}G(u).$$

Hence, reporting this equality in the expressions of (x_1, x_2, x_3) , we obtain a parametrization $u \mapsto \gamma(u)$ of this intersection.

Lemma 5.4. *On a curve where y_2 is constant we have*

$$\begin{aligned} y_1'(u) &= \frac{C^2 + G'^2}{C} \cos \varphi \cosh A, \\ y_3'(u) &= -\frac{C^2 + G'^2}{C} \sin \varphi \cosh A. \end{aligned}$$

Proof. Differentiating (16) we obtain $A' = C \cos^2 \varphi + \frac{\alpha}{C}G'$. Hence we get

$$\begin{aligned} y_1'(u) &= \frac{1}{\alpha} (G'' \cos \varphi - G' \varphi' \sin \varphi - C^2 \sin \varphi \cos^2 \varphi - \alpha G' \sin \varphi) \sinh A \\ &\quad + \frac{1}{\alpha} \left(C G' \cos^3 \varphi + \frac{\alpha}{C} G'^2 \cos \varphi - C \varphi' \cos \varphi \right) \cosh A \end{aligned}$$

and

$$\begin{aligned} y_3'(u) &= \frac{1}{\alpha} \left(\frac{C}{\alpha} G'' \cos \varphi - C \left(\frac{G'}{\alpha} - 1 \right) \varphi' \sin \varphi \right. \\ &\quad \left. - \left(\frac{C^2}{\alpha} + G' \right) \left(C \cos^2 \varphi + \frac{\alpha}{C} G' \right) \sin \varphi \right) \cosh A \\ &\quad + \frac{1}{\alpha} \left(C \left(\frac{G'}{\alpha} - 1 \right) \left(C \cos^2 \varphi + \frac{\alpha}{C} G' \right) \cos \varphi \right. \\ &\quad \left. - G'' \sin \varphi - \left(\frac{C^2}{\alpha} + G' \right) \varphi' \cos \varphi \right) \sinh A. \end{aligned}$$

We conclude using (9) and (13). \square

Proposition 5.5. *Let $c \in \mathbb{R}$. The intersection of \mathcal{C}_α and the vertical plane $\{y_2 = c\}$ is a non-empty closed embedded convex curve.*

Proof. This intersection is non-empty since setting $u = 0$ and $A = c$ gives $y_2 = c$. Also, by lemma 5.4 we have $y_1'^2 + y_3'^2 > 0$, so the intersection of \mathcal{C}_α and the vertical plane $\{y_2 = c\}$ is a smooth curve γ . Also, we have $\gamma(u + 2U) = \gamma(u)$, so the curve is closed.

We now prove that γ is embedded and convex. We consider the half of γ corresponding to $u \in (-\frac{U}{2}, \frac{U}{2})$. We have $\cos \varphi(u) > 0$. Then, by lemma 5.4, $u \mapsto y_1(u)$ is injective and increasing. We get

$$\frac{dy_3}{dy_1} = -\tan \varphi(u),$$

so $\frac{dy_3}{dy_1}$ is an increasing function of u , and also of y_1 . Consequently, the half of γ corresponding to $u \in (-\frac{U}{2}, \frac{U}{2})$ is an embedded convex arc and is situated below the segment linking its endpoints.

Finally, since $\gamma(u + U) = -\gamma(u)$, the whole curve is embedded and convex. \square

Theorem 5.6. *The horizontal catenoid \mathcal{C}_α has the following properties.*

1. *The intersection of \mathcal{C}_α and any vertical plane of equation $x_2 = c$ ($c \in \mathbb{R}$) is a non-empty closed embedded convex curve.*
2. *The surface \mathcal{C}_α is properly embedded.*
3. *The horizontal catenoid \mathcal{C}_α is invariant by rotations of angle π around the x_1 , x_2 and x_3 axes. The x_2 -axis is contained in the “interior” of \mathcal{C}_α .*
4. *It is conformally equivalent to $\mathbb{C} \setminus \{0\}$.*

Proof. 1. This is proposition 5.5.
 2. The fact that \mathcal{C}_α is embedded is a consequence of proposition 5.5. On a diverging path on \mathcal{C}_α , A must be diverging and so x_2 is diverging. Consequently, \mathcal{C}_α is proper.
 3. The symmetries of \mathcal{C}_α have already been proved. The x_2 -axis is contained in the “interior” of \mathcal{C}_α since each curve $x_2 = c$ ($c \in \mathbb{R}$) is convex and symmetric with respect to the x_2 -axis.
 4. The immersion $X = (F, h)$ induces a conformal bijective parametrization of \mathcal{C}_α by $\mathbb{C}/(\mathbb{Z}Z)$. \square

We now describe a few remarkable curves on \mathcal{C}_α .

The curve corresponding to $u = 0$ is the set of the lowest points of the curves $y_2 = c$ ($c \in \mathbb{R}$). This curve is given by

$$\begin{cases} y_1 &= \frac{\alpha - \sqrt{\alpha^2 + \cos(2\theta) - C^2}}{\alpha} \sinh\left(\frac{\alpha}{C} y_2\right), \\ y_3 &= -\frac{C \sqrt{\alpha^2 + \cos(2\theta) - C^2}}{\alpha} \cosh\left(\frac{\alpha}{C} y_2\right). \end{cases}$$

The curves along which \mathcal{C}_α is vertical correspond to $u = \pm \frac{U}{2}$ (because of formula (6)). They are symmetric one to the other with respect to the x_2 -axis. By (14) and (8), the curve corresponding to $u = \frac{U}{2}$ is given by

$$\begin{cases} y_1 &= \frac{C}{\alpha} \cosh\left(\frac{\alpha}{C} y_2\right), \\ y_3 &= \frac{2C^2 - \cos(2\theta)}{2\alpha^2} \sinh\left(\frac{\alpha}{C} y_2\right). \end{cases}$$

Consequently, the horizontal projection of \mathcal{C}_α is

$$\pi(\mathcal{C}_\alpha) = \left\{ (y_1, y_2) \in \mathbb{R}^2; |y_1| \leq \frac{C}{\alpha} \cosh\left(\frac{\alpha}{C} y_2\right) \right\}$$

It is a remarkable fact that this projection coincides with the projection of a minimal catenoid of \mathbb{R}^3 of parameter $\frac{C}{\alpha}$.

The curve given by $x_2 = 0$ is the analog of the “waist circle” of minimal catenoids in \mathbb{R}^3 .

Proposition 5.7. *On \mathcal{C}_α , there exists some points with negative curvature and some points with positive curvature. Moreover, \mathcal{C}_α has infinite total absolute curvature.*

Proof. Setting $\lambda = (G'^2 + C^2) \cosh^2 A$, the curvature of ds^2 is given by

$$K = -\frac{1}{2\lambda} \Delta_0(\ln \lambda)$$

where Δ_0 is the Laplacian with respect to $|dz|^2$. Thus we have

$$\begin{aligned} K\lambda &= -\frac{\partial}{\partial u}(\beta' \tanh A) - \frac{\partial}{\partial v}(\alpha \tanh A) - \frac{\partial}{\partial u} \left(\frac{G'G''}{G'^2 + C^2} \right) \\ &= 2C\varphi' \sin \varphi \cos \varphi \tanh A - \frac{C^2 \cos^4 \varphi + \alpha^2}{\cosh^2 A} \\ &\quad - (C^2 + G'^2) \sin^2 \varphi \cos^2 \varphi - G'\varphi'(2\cos^2 \varphi - 1) \end{aligned}$$

by (13).

Hence, when $u = \pm \frac{U}{2}$ we have $K\lambda = -\frac{\alpha^2}{\cosh^2 A} + \frac{\cos(2\theta)}{2}$, which is positive for $|A|$ large enough. On the other hand, when $u = A = 0$ we get $K\lambda = -2\alpha^2 - \cos(2\theta) + \alpha\sqrt{\alpha^2 + \cos(2\theta) - C^2} < 0$.

Finally, the total absolute curvature of \mathcal{C}_α is

$$\int_{-U}^U \int_{-\infty}^{+\infty} |K| \lambda du dv = +\infty$$

since, in general, $K\lambda$ does not tend to 0 when $v \rightarrow +\infty$ and u fixed. \square

6. LIMIT OF HORIZONTAL CATENOIDS AND VERTICAL HALF-SPACE THEOREM IN Nil_3

In this section we study the limit of \mathcal{C}_α when $\alpha \rightarrow +\infty$. As a corollary we obtain a vertical half-space theorem.

Since the parameter α will vary, the quantities and functions appearing in the construction of \mathcal{C}_α will be denoted by X_α , φ_α , U_α , C_α , β_α , etc. instead of X , φ , U , C , β , etc. They depend smoothly on α .

Proposition 6.1. *Let $(\hat{u}, \hat{v}) \in \mathbb{R}^2$. For $\alpha > 0$, let $u_\alpha = \frac{\hat{u}}{\alpha}$ and $v_\alpha = \frac{4 \ln \alpha + \hat{v}}{\alpha}$. Then, when $\alpha \rightarrow +\infty$,*

$$\begin{aligned} (y_1)_\alpha(u_\alpha, v_\alpha) &\rightarrow \frac{\sin \hat{u}}{4} e^{\hat{v}/2}, \\ (y_2)_\alpha(u_\alpha, v_\alpha) &\rightarrow 0, \\ (y_3)_\alpha(u_\alpha, v_\alpha) &\rightarrow -\frac{\cos \hat{u}}{4} e^{\hat{v}/2}. \end{aligned}$$

Proof. We have $\tilde{\theta}_\alpha \rightarrow \frac{\pi}{4}$ and so $C_\alpha \sim \frac{1}{2\alpha}$. We have $|(\beta_\alpha)'| \leq \frac{1}{2\alpha}$, thus $|\beta_\alpha(u)| \leq \frac{|u|}{2\alpha}$ and so $\beta_\alpha(u_\alpha) = O\left(\frac{1}{\alpha^2}\right)$. Also, for $\alpha \geq \frac{1}{2}$, we have

$$|(G_\alpha)'| = \left| \frac{C_\alpha^2 \cos^2 \varphi_\alpha - \cos(2\tilde{\theta}_\alpha)}{\alpha - (\varphi_\alpha)'} \right| \leq \frac{1}{4\alpha^3} + \frac{\cos(2\tilde{\theta}_\alpha)}{\alpha},$$

thus $G_\alpha(u_\alpha) = O\left(\frac{1}{\alpha^2}\right)$. From this we obtain that

$$(y_2)_\alpha(u_\alpha, v_\alpha) = C_\alpha v_\alpha - G_\alpha(u_\alpha) \rightarrow 0.$$

We also have

$$A = 4 \ln \alpha + \hat{v} + O\left(\frac{1}{\alpha^2}\right), \quad \cosh A \sim \frac{1}{2} e^{\hat{v}/2} \alpha^2, \quad \sinh A \sim \frac{1}{2} e^{\hat{v}/2} \alpha^2.$$

And, since, for $\alpha \geq 1$, $-\sqrt{\alpha^2 + 1} \leq (\varphi_\alpha)' \leq -\sqrt{\alpha^2 - 1}$, we have $\varphi_\alpha(u_\alpha) \rightarrow -\hat{u}$. This concludes the proof. \square

This proposition means that, when $\alpha \rightarrow +\infty$, the half of \mathcal{C}_α corresponding to $A > 0$ converges to the punctured vertical plane $\{x_2 = 0\} \setminus \{(0, 0, 0)\}$. In the same way one can prove that the other half of \mathcal{C}_α converges to this punctured vertical plane.

Lemma 6.2. *The curve of equation $y_2 = 0$ in \mathcal{C}_α converges uniformly to 0 when $\alpha \rightarrow +\infty$.*

Proof. On the curve of equation $y_2 = 0$ in \mathcal{C}_α we have

$$|(y_1)_\alpha(u)| \leq \frac{|G'_\alpha(u)|}{\alpha} \sinh |A_\alpha(u)| + \frac{C_\alpha}{\alpha} \cosh A_\alpha(u),$$

$$|(y_3)_\alpha(u)| \leq \frac{C_\alpha}{\alpha} \left(\frac{|G'_\alpha(u)|}{\alpha} + 1 \right) \cosh A_\alpha(u) + \frac{1}{\alpha} \left(\frac{C_\alpha^2}{\alpha} + |G'_\alpha(u)| \right) \sinh |A_\alpha(u)|,$$

with

$$A_\alpha(u) = \beta_\alpha(u) + \frac{\alpha}{C_\alpha} G_\alpha(u).$$

By the computations done in the proof of proposition 6.1 we have, for $\alpha \geq 1$, $|G'_\alpha(u)| \leq \frac{2}{\alpha}$ and $C_\alpha \leq \frac{1}{2\alpha}$. Hence it suffices to prove that A_α is uniformly bounded when $\alpha \rightarrow +\infty$.

By (14), the function $\beta_\alpha + \frac{\alpha}{C_\alpha} G_\alpha$ is $2U_\alpha$ -periodic with

$$U_\alpha = \int_0^{U_\alpha} du = \int_{-1}^1 \frac{dx}{\sqrt{(1-x^2)P_\alpha(x)}}.$$

We now assume that $\alpha \leq 1$. For $x \in [-1, 1]$ we have $P_\alpha(x) \geq \alpha^2 - 1$, and so $U_\alpha \leq \frac{\pi}{\sqrt{\alpha^2 - 1}}$. Using the bounds on β'_α and G'_α and the periodicity we get

$$|A_\alpha(u)| \leq \frac{\pi}{2\alpha\sqrt{\alpha^2 - 1}} + \frac{\alpha}{C_\alpha} \frac{2\pi}{\alpha\sqrt{\alpha^2 - 1}}.$$

Next, since $C_\alpha \sim \frac{1}{2\alpha}$, we conclude that A_α is uniformly bounded when $\alpha \rightarrow +\infty$, which ends the proof. \square

Theorem 6.3 (vertical half-space theorem). *Let Σ be a properly immersed minimal surface in Nil_3 . Assume that Σ is contained on the one side of a vertical plane P . Then Σ is a vertical plane parallel to P .*

Proof. We assume that Σ is not a vertical plane.

We proceed as in [HM90]. Up to an isometry of Nil_3 we can assume that P is the plane $\{y_2 = 0\}$, that $\Sigma \subset \{y_2 \leq 0\}$ and that Σ is not contained in any half-space $\{y_2 \leq -\varepsilon\}$ for $\varepsilon > 0$. By the maximum principle, we necessarily have $\Sigma \cap P = \emptyset$.

We use the coordinates (y_1, y_2, y_3) defined by (15). For $\varepsilon \in \mathbb{R}$, let $T_\varepsilon : (y_1, y_2, y_3) \mapsto (y_1, y_2 + \varepsilon, y_3)$ (this is a translation in the y_2 direction, an isometry of Nil_3). Then, for $\varepsilon > 0$ sufficiently small, we have $T_\varepsilon(\Sigma) \cap P \neq \emptyset$.

For $\alpha \geq 1$ we consider the half-horizontal catenoid $\mathcal{C}'_\alpha = \mathcal{C}_\alpha \cap \{y_2 \geq 0\}$. By lemma 6.2, there exists a compact subset \mathcal{D} of P containing 0 and $\mathcal{C}_\alpha \cap P$ for all $\alpha \geq 1$.

We claim that there exists $\varepsilon > 0$ such that

$$T_\varepsilon(\Sigma) \cap P \neq \emptyset, \quad T_\varepsilon(\Sigma) \cap \mathcal{C}'_1 = \emptyset, \quad T_\varepsilon(\Sigma) \cap \mathcal{D} = \emptyset.$$

Assume the claim is false. Since $T_\eta(\Sigma) \cap P \neq \emptyset$ for η small enough, this means that there exists a sequence (ε_n) of positive numbers converging to 0 and a sequence (q_n) of points such that $q_n \in T_{\varepsilon_n}(\Sigma)$ and $q_n \in \mathcal{C}'_1 \cup \mathcal{D}$ for all n . In particular, for n large enough, q_n belongs to the union of \mathcal{D} and the part of \mathcal{C}'_1 between the planes $\{y_2 = 0\}$ and $\{y_2 = 1\}$, which is compact. Hence, up to extraction of a subsequence, we can assume that q_n converges to a point q . We necessarily have $q \in P$, and, since Σ is proper, $q \in \Sigma$. This contradicts the fact that $\Sigma \cap P = \emptyset$, which proves the claim.

By proposition 6.1, \mathcal{C}'_α converges smoothly, away from 0, to $P \setminus \{0\}$ when $\alpha \rightarrow +\infty$. Hence, for α large enough, $\mathcal{C}'_\alpha \cap T_\varepsilon(\Sigma) \neq \emptyset$. Also, by continuity of the family (\mathcal{C}_α) , we have $\mathcal{C}'_\alpha \cap T_\varepsilon(\Sigma) = \emptyset$ for α close enough to 1.

Let $\Gamma = \{\alpha \geq 1; \mathcal{C}'_\alpha \cap T_\varepsilon(\Sigma) \neq \emptyset\}$ and $\gamma = \inf \Gamma$. We have $\gamma > 1$. We claim that $\gamma \in \Gamma$.

If γ is an isolated point, then it is clear. We now assume that γ is not isolated. Then there exists a decreasing sequence (α_n) converging to γ and a sequence of points (p_n) such that $p_n \in \mathcal{C}'_{\alpha_n} \cap T_\varepsilon(\Sigma)$. We can write $p_n = X_{\alpha_n}(u_n, A_n)$ with $u_n \in [-U_{\alpha_n}, U_{\alpha_n}]$ and $A_n \in \mathbb{R}$. We have $0 \leq (y_2)_{\alpha_n}(p_n) \leq \varepsilon$, i.e.,

$$0 \leq \frac{C_{\alpha_n}}{\alpha_n} A_n - \frac{C_{\alpha_n}}{\alpha_n} \beta_{\alpha_n}(u_n) - G_{\alpha_n}(u_n) \leq \varepsilon.$$

Since for all n we have $\alpha_n \in [\gamma, \alpha_0]$, u_n is bounded and so $|\frac{C_{\alpha_n}}{\alpha_n} \beta_{\alpha_n}(u_n) + G_{\alpha_n}(u_n)|$ is also bounded; moreover C_{α_n} is bounded from below by a positive constant. From this we deduce that A_n is bounded. Consequently, up to extraction of a subsequence, we can assume that (u_n, A_n) converges to some $(u, A) \in \mathbb{R}$. Then, by continuity, p_n converges to a point lying in $T_\varepsilon(\Sigma)$ and in \mathcal{C}'_γ . This finishes proving the claim.

Thus there exists a point $p \in \mathcal{C}'_\gamma \cap T_\varepsilon(\Sigma)$. Since $\partial \mathcal{C}'_\gamma \subset \mathcal{D}$ (by construction of \mathcal{D}) and $T_\varepsilon(\Sigma) \cap \mathcal{D} = \emptyset$, p is an interior point of \mathcal{C}'_γ . Moreover, since $\mathcal{C}'_\alpha \cap T_\varepsilon(\Sigma) = \emptyset$ for all $\alpha < \gamma$, \mathcal{C}'_γ lies on one side of $T_\varepsilon(\Sigma)$ in a neighbourhood of p . Then, by the maximum principle we get $T_\varepsilon(\Sigma) = \mathcal{C}_\gamma$, which gives a contradiction since a horizontal catenoid is not contained in a half-space. \square

Remark 6.4. Apart from the fact that horizontal catenoids converge to a punctured vertical plane, the key fact in this proof is that horizontal catenoids meet all vertical planes $\{y_2 = c\}$ for $c \in \mathbb{R}$ (proposition 5.5). This ensures that the sequence (p_n) is bounded.

For example, for minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$, rotational catenoids have finite height (see [NR02]), and so there is no half-space theorems with respect to horizontal planes.

Remark 6.5. Abresch and Rosenberg [AR05] proved a half-space theorem with respect to surfaces of equation $x_3 = c$ ($c \in \mathbb{R}$). It relies on the fact that rotational catenoids converge to such a surface.

7. HELICOIDAL MINIMAL SURFACES

In this section we investigate the minimal surface \mathcal{H}_α in Nil_3 whose Gauss map is the function g defined in section 4, with $\theta = 0$. The calculus of Proposition 5.1 still holds with $C = 0$, $\beta = 0$ and $A = \alpha v$.

The coordinates (y_1, y_2, y_3) of \mathcal{H}_α are

$$\begin{aligned} y_1 &= \frac{G'(u)}{\alpha} \cos \varphi(u) \sinh(\alpha v), \\ y_2 &= -G(u), \\ y_3 &= -\frac{G'(u)}{\alpha} \sin \varphi(u) \sinh(\alpha v). \end{aligned}$$

In particular we have $\frac{y_3}{y_1} = -\tan \varphi(u)$ and y_2 only depends on u . This means that the intersection of \mathcal{H}_α and any vertical plane $\{y_2 = c\}$ is a straight line. Moreover the surface is simply periodic since φ is periodic.

8. A FAMILY OF HORIZONTAL CMC $\frac{1}{2}$ ANNULI

In this section we integrate the equations of Fernandez and Mira to construct a one-parameter family of horizontal annuli. These surfaces are the sister surfaces of the helicoidal surfaces construct in section 7.

We consider the harmonic map $g : \Sigma \rightarrow \mathbb{H}^2$ given in section 4, with $\theta = 0$ and g_* the conjugate harmonic map (see section 4).

Lemma 8.1. *The following harmonic maps are conjugate:*

$$\begin{aligned} g &= \frac{\sin \varphi(u) + i \sinh(\alpha v)}{\cos \varphi(u) + \cosh(\alpha v)}, \\ g_* &= \frac{\sin \varphi_*(u) + i \sinh(\alpha_* v)}{\cos \varphi_*(u) + \cosh(\alpha_* v)}, \end{aligned}$$

with $\varphi'^2 - \alpha^2 = \cos^2 \varphi$, $\varphi(0) = 0$ and $\varphi_*'^2 - \alpha_*^2 = -\cos^2 \varphi_*$, $\varphi_*(0) = 0$ and $\alpha_*^2 = \alpha^2 + 1$.

Proof. We remark that g_* is harmonic as g in section 4. Moreover $Q(g_*) = -\frac{1}{4}dz^2$. The conformal minimal immersions in $\mathbb{H}^2 \times \mathbb{R}$ are given by $Y(u, v) = (g(u, v), v)$ and $Y_* = (g_*(u, v), u)$. To be isometric, it suffices to check that $\cosh \omega = \cosh \omega_*$, i.e.,

$$\cosh^2 \omega = \frac{4|g_u|^2}{(1 - |g|^2)^2} = 1 + \frac{4|g_v|^2}{(1 - |g|^2)^2} = \frac{4|g_{*v}|^2}{(1 - |g_*|^2)^2} = 1 + \frac{4|g_{*u}|^2}{(1 - |g_*|^2)^2}$$

These relations are equivalent to

$$\frac{\varphi'^2}{\cos^2 \varphi} = 1 + \frac{\alpha^2}{\cos^2 \varphi} = 1 + \frac{\varphi_*'^2}{\cos^2 \varphi_*} = \frac{\alpha_*^2}{\cos^2 \varphi_*}.$$

A straightforward computation shows the functions $\frac{\varphi'}{\cos \varphi}$ and $\frac{\alpha_*}{\cos \varphi_*}$ are both solutions of

$$A'^2 = (A^2 - 1)(A^2 - \alpha^2 - 1).$$

Moreover we have $\varphi(0) = \varphi_*(0) = 0$ because $\alpha_*^2 = \alpha^2 + 1$. This concludes the proof. \square

In summary we have $Q(g) = \frac{1}{4}dz^2 = -Q(g_*)$ and $\tau = \tau^* = e^{2\omega}$. The map g induces locally a minimal graph in Nil_3 by theorem 2.3, with metric

$$\lambda = \frac{\tau}{\nu^2}|dz|^2 = 16 \frac{(1+|g|^2)^2}{(1-|g|^2)^4}|g_z|^2|dz|^2.$$

Then

$$(17) \quad \nu^2 = \frac{1-|g|^2}{1+|g|^2} = \frac{\cos^2 \varphi}{\cosh^2(\alpha v)}, \quad \tau = \frac{(\varphi' + \alpha)^2}{\cos^2 \varphi} = \frac{(\varphi'_* + \alpha_*)^2}{\cos^2 \varphi_*}.$$

This minimal multigraph is isometric to an immersed CMC $\frac{1}{2}$ surface in $\mathbb{H}^2 \times \mathbb{R}$ with harmonic Gauss map $g_* : \Sigma \rightarrow \mathbb{H}^2$ admitting data $(-Q, \tau)$. For $a_0 \in \mathbb{C}$, there is a unique solution h^* of the following system

$$\begin{cases} h_{zz}^* = (\log \tau)_z h_z^* + Q \sqrt{\frac{\tau + 4|h_z^*|^2}{\tau}} \\ h_{z\bar{z}}^* = \frac{1}{4} \sqrt{\tau(\tau + 4|h_z^*|^2)} \\ h_z^*(z_0) = a_0 \end{cases}$$

with $\tau + 4|h_z^*|^2 = \lambda$, and using (17), $\varphi'' + \sin \varphi \cos \varphi = 0$, $Q = \frac{1}{4}$ we obtain

$$\begin{cases} h_{zz}^* = \alpha \tan \varphi h_z^* + \frac{\cosh(\alpha v)}{4 \cos \varphi} \\ h_{z\bar{z}}^* = \frac{(\varphi' + \alpha)^2 \cosh(\alpha v)}{4 \cos^3 \varphi} \end{cases}$$

Now set $H = h_z^*$, then

$$\begin{cases} H_z = \alpha \tan \varphi H + \frac{\cosh(\alpha v)}{4 \cos \varphi} \\ H_{\bar{z}} = \frac{(\varphi' + \alpha)^2 \cosh(\alpha v)}{4 \cos^3 \varphi} \\ H(z_0) = a_0. \end{cases}$$

Then

$$H(u, v) = \frac{\cos \varphi}{2(\alpha - \varphi')} (i \sinh(\alpha v) - \tan \varphi \cosh(\alpha v)) + K_1(u) e^{i(\alpha \tan \varphi)v} + K_2(u)$$

where $K'_i = a \tan \varphi K_i$ and $K_1(u_0), K_2(u_0)$ are chosen to have $H(z_0) = a_0$. It is a two-parameter family and we are interested in a_0 such that, $K_1 = K_2 = 0$, i.e., the solution with $\tau + 4|H|^2 = \lambda$. The solution is periodic in u and by (17) we have

$$h^* = \frac{\cos \varphi \cosh(\alpha v)}{\alpha(\varphi' - \alpha)} = \frac{\cos \varphi_* \cosh(\alpha v)}{\alpha(\varphi'_* - \alpha_*)}.$$

Now we consider

$$(G_1, G_2, G_3) = \left(\frac{2g_*}{1-|g_*|^2}, \frac{1+|g_*|^2}{1-|g_*|^2} \right) = \left(\tan \varphi_*, \frac{\sinh(\alpha_* v)}{\cos \varphi_*}, \frac{\cosh(\alpha_* v)}{\cos \varphi_*} \right)$$

and we compute the horizontal component $F^* = \frac{X_1 + iX_2}{1 + X_3}$ given by

$$X_j = \frac{8 \operatorname{Re}(G_{j,z}(4\bar{Q}h_z + \tau h_{\bar{z}}))}{\tau^2 - 16|Q|^2} + G_j \sqrt{\frac{\tau + 4|h_z|^2}{\tau}}$$

i.e.,

$$X_j = (\alpha_* - \varphi'_*) \left(\frac{G_{j,u} h_u}{\varphi'_*} + \frac{G_{j,v} h_v}{\alpha_*} \right) + G_j \frac{\cosh(\alpha v)}{\cos \varphi}.$$

Straightforward computations give

$$\begin{aligned} h_u &= \frac{\alpha_* \sin \varphi_* \cosh(\alpha v)}{\alpha(\varphi'_* - \alpha_*)} & h_v &= \frac{\cos \varphi_* \sinh(\alpha v)}{(\varphi'_* - \alpha_*)} \\ G_{1,u} &= \frac{\varphi'_*}{\cos^2 \varphi_*} & G_{1,v} &= 0 \\ G_{2,u} &= \frac{\varphi'_* \sin \varphi_* \sinh(\alpha_* v)}{\cos^2 \varphi_*} & G_{2,v} &= \frac{\alpha_* \cosh(\alpha_* v)}{\cos \varphi_*} \\ G_{3,u} &= \frac{\varphi'_* \sin \varphi_* \cosh(\alpha_* v)}{\cos^2 \varphi_*} & G_{3,v} &= \frac{\alpha_* \sinh(\alpha_* v)}{\cos \varphi_*} \end{aligned}$$

Inserting the explicit value above, setting $f(u) = \frac{\alpha \cos \varphi_* - \alpha_* \cos \varphi}{\alpha \cos \varphi \cos^2 \varphi_*}$, we obtain

$$\begin{cases} X_1 = \cosh(\alpha v) \sin \varphi_*(u) f(u) \\ X_2 = \cosh(\alpha v) \sinh(\alpha_* v) \left(f(u) + \frac{\alpha_*}{\alpha} \right) - \cosh(\alpha_* v) \sinh(\alpha v) \\ X_3 = \cosh(\alpha v) \cosh(\alpha_* v) \left(f(u) + \frac{\alpha_*}{\alpha} \right) - \sinh(\alpha_* v) \sinh(\alpha v) \end{cases}$$

Now we are interested in level curves at height zero. Then by (8), we have $\varphi(-U/2) = \varphi_*(-U/2) = +\pi/2$ and $\varphi(U/2) = \varphi_*(U/2) = -\pi/2$. We have $h(-U/2, v) = h(U/2, v) = 0$. Using the ODE of φ and φ_* we obtain $f(u) \rightarrow \gamma = \frac{-1}{2\alpha\alpha_*}$ when $u \rightarrow \pm U/2$. Then the horizontal curve $h^* = 0$ has two connected components given by $F^*(\pm U/2, v) = \frac{X_1 + iX_2}{1 + X_3}$ with

$$\begin{cases} X_1 = \cosh(\alpha v) \sin \varphi_*(\pm U/2) \gamma \\ X_2 = \cosh(\alpha v) \sinh(\alpha_* v) \left(\gamma + \frac{\alpha_*}{\alpha} \right) - \cosh(\alpha_* v) \sinh(\alpha v) \\ X_3 = \cosh(\alpha v) \cosh(\alpha_* v) \left(\gamma + \frac{\alpha_*}{\alpha} \right) - \sinh(\alpha_* v) \sinh(\alpha v) \end{cases}$$

We remark that $\gamma + \frac{\alpha_*}{\alpha} = \frac{2\alpha^2 + 1}{2\alpha\alpha_*} \geq 1$. In the disk model $F^*(-U/2, v)$ and $F^*(U/2, v)$ are symmetric with respect to the y -axis of the disk ($X_1(-U/2, v) = -X_1(U/2, v)$). Then we will study $F^*(-U/2, v)$. It is a curve linking the point $(0, -1)$ to $(0, 1)$ in the unit disk and staying in $\text{Re } F^* > 0$. We prove that this curve is embedded and behaves like a generatrix of a Bryant catenoid in hyperbolic three-space. In the half-plane model of \mathbb{H}^2 , this curve is given by

$$\begin{aligned} \tilde{F}(-U/2, v) &= \left(\frac{X_1}{X_3 - X_2}, \frac{1}{X_3 - X_2} \right) \\ &= \left(\frac{\gamma e^{\alpha_* v}}{(\gamma + \frac{\alpha_*}{\alpha} + \tanh(\alpha v))}, \frac{e^{\alpha_* v}}{\cosh(\alpha v)(\gamma + \frac{\alpha_*}{\alpha} + \tanh(\alpha v))} \right). \end{aligned}$$

By a straightforward computation we can see that the map

$$v \mapsto \frac{\gamma e^{\alpha_* v}}{(\gamma + \frac{\alpha_*}{\alpha} + \tanh(\alpha v))}$$

is strictly increasing for $\alpha \leq 1$ and has exactly one point where the derivative is zero when $\alpha = 1$. When $\alpha \geq 1$ the function has two local extrema at points v where $\tanh(\alpha v_{\pm}) = \frac{-\sqrt{\alpha^2+1} \pm \sqrt{\alpha^2-1}}{2\alpha} < 0$. For $v > 0$, $v(x_1)$ is a well defined function and $\frac{1}{X_3 - X_2} = x_1^{1-\frac{\alpha}{\alpha^*}} q(x_1)$ where $q(x_1)$ is a bounded function having a positive limit at infinity. When $\alpha \rightarrow \infty$, the curve converges to two tangent horocycle.

The immersion for $u \in [-U/2, U/2]$ is a graph over a simply connected domain of \mathbb{H}^2 . We complete it by reflection about the horizontal plane of height zero in $\mathbb{H}^2 \times \mathbb{R}$ to obtain a properly embedded annulus.

9. APPENDIX: PROOFS OF LEMMAS 4.5 AND LEMMA 4.6

We use the notations of section 4. We notice that, for $(\alpha, \theta) \in \Omega$, $\frac{-\alpha\sqrt{P_{\alpha,\theta}(x)+\alpha^2}}{x^2}$ can be extended smoothly at $x = 0$ and that

$$L(\alpha, \theta) = \int_{-1}^1 \frac{-\alpha\sqrt{P_{\alpha,\theta}(x)} + \alpha^2 + C_{\alpha,\theta}^2 x^4}{x^2 \sqrt{(1-x^2)} \sqrt{P_{\alpha,\theta}(x)}} dx.$$

Lemma (lemma 4.5). *Let $\alpha > 0$. Then there exists a unique $\tilde{\theta}_\alpha \in (0, \theta_\alpha^+) \cap (0, \frac{\pi}{4})$ such that*

$$L(\alpha, \tilde{\theta}_\alpha) = 0.$$

Proof. We have

$$L(\alpha, \theta) = \int_{-1}^1 \frac{l(\alpha, \theta, x)}{\sqrt{1-x^2}} dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} l(\alpha, \theta, \sin t) dt$$

where l is a smooth function on $\Omega \times [-1, 1]$. Hence L is smooth on Ω .

We have $L(\alpha, 0) < 0$, since $C_{\alpha,0} = 0$.

We first deal with the case where $\alpha > \frac{1}{\sqrt{2}}$, i.e., $\theta_\alpha^+ > \frac{\pi}{4}$. Since the integrand in $L(\alpha, \frac{\pi}{4})$ is positive for all $x \in (-1, 1)$, we have $L(\alpha, \frac{\pi}{4}) > 0$. Hence, by continuity, there exists $\tilde{\theta}_\alpha \in (0, \frac{\pi}{4})$ such that $L(\alpha, \tilde{\theta}_\alpha) = 0$.

We now deal with the case where $\alpha \leq \frac{1}{\sqrt{2}}$, i.e., $\theta_\alpha^+ \leq \frac{\pi}{4}$. We have $C_{\alpha,\theta}^2 \rightarrow 1 - \alpha^2$, $\rho_{\alpha,\theta}^- \rightarrow 1$ and $\rho_{\alpha,\theta}^+ \rightarrow \frac{\alpha^2}{1-\alpha^2} > 0$ when $\theta \rightarrow \theta_\alpha^+$. We have

$$L(\alpha, \theta) = L_1(\alpha, \theta) + L_2(\alpha, \theta)$$

with

$$\begin{aligned} L_1(\alpha, \theta) &= \int_{-1}^1 -\frac{2\alpha C_{\alpha,\theta}^2}{\alpha + \sqrt{P_{\alpha,\theta}(x)}} \sqrt{\frac{1-x^2}{P_{\alpha,\theta}(x)}} dx, \\ L_2(\alpha, \theta) &= \int_{-1}^1 \frac{2\alpha C_{\alpha,\theta}^2 - \alpha \cos(2\theta) + C_{\alpha,\theta}^2 x^2 \sqrt{P_{\alpha,\theta}(x)}}{\sqrt{(1-x^2)} P_{\alpha,\theta}(x) (\alpha + \sqrt{P_{\alpha,\theta}(x)})} dx. \end{aligned}$$

We claim that $\frac{1-x^2}{P_{\alpha,\theta}(x)}$ is uniformly bounded (in x) when $\theta \rightarrow \theta_\alpha^+$. Indeed we have

$$\begin{aligned} \left| \frac{1-x^2}{P_{\alpha,\theta}(x)} - \frac{1}{\alpha^2 + (1-\alpha^2)x^2} \right| &= \left| \frac{(1-2\alpha^2 - \cos(2\theta))x^2 - (1-\alpha^2 - C_{\alpha,\theta}^2)x^4}{C_{\alpha,\theta}^2(\rho_{\alpha,\theta}^- - x^2)(\rho_{\alpha,\theta}^+ + x^2)(\alpha^2 + (1-\alpha^2)x^2)} \right| \\ &\leq \frac{|1-2\alpha^2 - \cos(2\theta)| + |1-\alpha^2 - C_{\alpha,\theta}^2|}{C_{\alpha,\theta}^2(\rho_{\alpha,\theta}^- - 1)\rho_{\alpha,\theta}^+ \alpha^2} \\ &\leq \frac{(\cos(2\theta) + 1 + 2\alpha^2)(1 - \cos(2\theta))}{4\alpha^2 C_{\alpha,\theta}^2 \rho_{\alpha,\theta}^+}. \end{aligned}$$

This upper bound has a finite limit when $\theta \rightarrow \theta_\alpha^+$. This proves the claim. Consequently, $L_1(\alpha, \theta)$ is bounded when $\theta \rightarrow \theta_\alpha^+$. Moreover we have $2\alpha C_{\alpha, \theta}^2 - \alpha \cos(2\theta) \rightarrow \alpha$ when $\theta \rightarrow \theta_\alpha^+$, so there exists a positive constant c_α such that, for θ close enough to θ_α^+ ,

$$L_2(\alpha, \theta) \geq \int_{-1}^1 \frac{c_\alpha}{\sqrt{(1-x^2)(\rho_{\alpha, \theta}^- - x^2)}}.$$

Since $\rho_{\alpha, \theta}^- \rightarrow 1$ when $\theta \rightarrow \theta_\alpha^+$, we obtain that $L_2(\alpha, \theta) \rightarrow +\infty$ when $\theta \rightarrow \theta_\alpha^+$. We conclude that $L(\alpha, \theta) \rightarrow +\infty$ when $\theta \rightarrow \theta_\alpha^+$. Hence, by continuity, there exists $\tilde{\theta}_\alpha \in (0, \theta_0)$ such that $L(\alpha, \tilde{\theta}_\alpha) = 0$.

To prove the uniqueness of $\tilde{\theta}_\alpha$, it suffices to prove that $\theta \mapsto L(\alpha, \theta)$ is increasing on $(0, \theta_\alpha^+) \cap (0, \frac{\pi}{4})$. A straightforward computation gives

$$\frac{\partial}{\partial \theta} \left(\frac{-\alpha \sqrt{P_{\alpha, \theta}(x)} + \alpha^2 + C_{\alpha, \theta}^2 x^4}{x^2 \sqrt{P_{\alpha, \theta}(x)}} \right) = \frac{C_{\alpha, \theta}}{\alpha} \frac{B_1}{P_{\alpha, \theta}(x)^{\frac{3}{2}}}$$

with

$$B_1 = 2 \cos(2\theta) x^2 P_{\alpha, \theta}(x) + (2\alpha^2 + \cos(2\theta) x^2)(\alpha^2 + C_{\alpha, \theta}^2 x^4) > 0.$$

This shows that $L(\alpha, \theta)$ is increasing in θ , which proves the uniqueness of $\tilde{\theta}_\alpha$. \square

Lemma (lemma 4.6). *We have*

$$\lim_{\alpha \rightarrow +\infty} \tilde{\theta}_\alpha = \frac{\pi}{4}.$$

Proof. We set $C_\alpha = C_{\alpha, \tilde{\theta}_\alpha}$ and $P_\alpha(x) = P_{\alpha, \tilde{\theta}_\alpha}(x)$. We first notice that $C_\alpha \leq \frac{1}{2\alpha}$ and, for $\alpha \geq \frac{1}{2}$ and $x \in [-1, 1]$, $\alpha^2 - 1 \leq P_\alpha(x) \leq \alpha^2 + 1$.

We have

$$0 = \alpha L(\alpha, \tilde{\theta}_\alpha) = I_1(\alpha) - \cos(2\tilde{\theta}_\alpha) I_2(\alpha) + I_3(\alpha)$$

with

$$\begin{aligned} I_1(\alpha) &= \int_{-1}^1 \frac{2\alpha^2 C_\alpha^2}{\sqrt{P_\alpha(x)}(\alpha + \sqrt{P_\alpha(x)})} \frac{x^2 dx}{\sqrt{1-x^2}} = O\left(\frac{1}{\alpha^2}\right), \\ I_2(\alpha) &= \int_{-1}^1 \frac{\alpha^2}{\sqrt{P_\alpha(x)}(\alpha + \sqrt{P_\alpha(x)})} \frac{dx}{\sqrt{1-x^2}} \geq \frac{\pi \alpha^2}{\sqrt{\alpha^2 + 1}(\alpha + \sqrt{\alpha^2 + 1})}, \\ I_3(\alpha) &= \int_{-1}^1 \frac{\alpha C_\alpha^2}{\alpha + \sqrt{P_\alpha(x)}} \frac{x^2 dx}{\sqrt{1-x^2}} = O\left(\frac{1}{\alpha^2}\right). \end{aligned}$$

Hence $\cos(2\tilde{\theta}_\alpha) \rightarrow 0$, which proves the lemma. \square

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